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Short Papers

On a Direct Use of Edge Condition in Modal Analysis

C. VASSALLO

***Abstract*—The edge condition allows us to know the asymptotic decrease of modal amplitudes in some discontinuity problems in waveguides. One may take a direct account of this information in modal analysis and gain a significant improvement of the calculation when the field singularity at edge is important. The accuracy and the validity of this method are studied in two cases: the diaphragm and the junction between an empty waveguide and a partially dielectric-filled waveguide.**

Introduction

The modal analysis is appropriate for all the waveguide discontinuities contained in a single cross-section plane, i.e., discontinuities like irises or abrupt transitions from one kind of guide to another one [1]. Its formulation is very easy, and modern computers can cope with the high-rank linear systems which may result from its application. However, these systems are only the truncated approximations of the theoretical systems of infinite rank in a rigorous formulation of the method, and some difficulties, such as the relative convergence effect, may lead to false results [2], [3] or a too slow rate of convergence may lead to inaccurate results. In this work we present a method based upon the edge effect theory, which may improve the convergence. We shall present our method in Section I, then we shall study its application to different kinds of discontinuities in order to know its range of interest.

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I. Modal Analysis and Edge Effect

Let us consider an abrupt transition between a left waveguide the kth normal mode of which has transverse components (e_k', h_k'), and a right waveguide the pth normal mode of which has transverse components (e_p'', h_p''). The equations which describe the scattering of the nth left normal mode on the transition have the following form:

$$\sum_k (\delta_{kn} + R_k) e_k'(x,y) = \sum_p T_p e_p''(x,y) \qquad (1)$$

$$\sum_k (\delta_{kn} - R_k) h_k'(x,y) = \sum_p T_p h''(x,y) \qquad (2)$$

where the unknown coefficients are (R_k) and (T_p) $(k,p = 1,2,\cdots)$. By taking the cross product of the two sides of these equations with the functions of any set complete on the cross section, one obtains an equivalent infinite algebraic linear system. For instance, with the set $\{e_p''\}$ one may transform (1) into

$$T_p = \sum_{k=1}^{\infty} (\delta_{kn} + R_k) V_{pk}, \qquad (p = 1,2,\cdots,\infty) \qquad (3)$$

where the V_{pk} are defined by integrals on the mode components. Equation (2) is transformed in a similar way.

The practical resolution consists of retaining a finite number of unknown modal coefficients. For instance, system (3) is replaced by

$$T_p = \sum_{k=1}^{N} (\delta_{kn} + R_k) V_{pk}, \qquad (p = 1,2,\cdots,P) \qquad (4)$$

and the integers N and P are chosen in order to have as many equations as unknown coefficients. Then, one has an ordinary linear system.

In that process one gives up any available knowledge on the modal coefficients of higher order. Now, such a knowledge exists as soon as there is an edge effect in the discontinuity between the two guides; then, one may write $R_k \simeq \bar{R}f(k)$ for high k, where $\bar{R}$ is unknown and $f(k)$ is a known function (Appendix A). It is possible to introduce this information in the computation by replacing (4) by

$$T_p = \sum_{k=1}^{N} V_{pk}(\delta_{kn} + R_k) + \bar{R} \sum_{k>N} V_{pk} f(k). \qquad (5)$$

There is one more unknown coefficient ($\bar{R}$), and thus one has to retain one more value for integer p. It seems reasonable to think that (5) is a better approximation of (3) than the mere truncation of (4), and we may hope for a better behavior of the calculations with this new system.

In the following sections of this short paper we present the results of our method in different cases of edge effect; we intend to study its accuracy in comparison with the usual method. Section II deals with capacitive and inductive diaphragms and Section III deals with dielectric steps.

II. Diaphragms

In order to study the convergence properties of our method, it is better to deal with a problem which has an analytical solution. We chose the asymmetrical semidiaphragm in rectangular waveguide, either capacitive or inductive (Fig. 1), and we calculated its equivalent shunt impedance by different ways.

1) Exact calculation.[1]

2) Modal analysis with ordinary truncation. Different methods are possible; we retained that described by Lee, Jones, and Campbell (LJC) [2], which has a quite better rate of convergence than that described by Mittra, Itoh, and Li (MIL) [3]. These different methods correspond to the use of different complete sets to transform (1) and (2).

3) Our method with edge effect. Its only difficulty lies in the summation of the series appearing in (5) and it depends strongly on the formalism followed before truncation. For instance, the LJC method leads to a series like $\sum_k k^{-m} \sin(k\alpha\pi)$ where α is the relative width of the diaphragm; there is no problem with semidiaphragm ($\alpha = 1/2$) but there would be one with arbitrary α. We preferred the MIL method for our calculations; the corresponding series behaves as $\sum k^{-m}$ whatever α is.

In Figs. 2 and 3 we have plotted the relative error on shunt impedance $(X - X_{\text{exact}})/X_{\text{exact}}$ versus the number of modes retained in the truncation, both with usual modal analysis and with our method, for three frequencies: just above the cutoff of the dominant mode, just below the cutoff of the first higher mode, and an intermediate frequency. One can see that taking account of the edge effect leads to a substantial improvement in accuracy, especially for the capacitive diaphragm. The improvement is less spectacular for the inductive diaphragm; we think that must be connected with the nature of the field singularity which is quite weaker in the second case (singularity in $r^{1/2}$ instead of $r^{-1/2}$ for the capacitive case) [4, pp. 18–20].

We have reported similar results elsewhere on the problem of bifurcation [5], which also has an analytical solution and may be connected to the problem of the diaphragm [3]. Especially, we found that there is no longer any relative convergence effect [2], [3] with our method, which may be easily understood since that effect is due to the truncation in usual modal analysis while in our method the higher terms are retained, at least in their asymptotic form.

[1] See [4, p. 441] for inductive semidiaphragm and [4, pp. 347 and 452] for capacitive semidiaphragms.

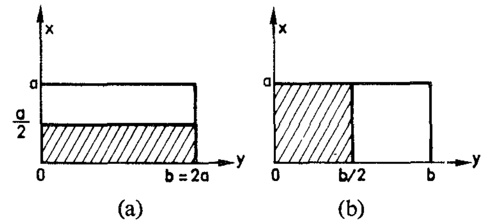

Fig. 1. (a) Capacitive semidiaphragm in rectangular waveguide. (b) Inductive semidiaphragm in rectangular waveguide.

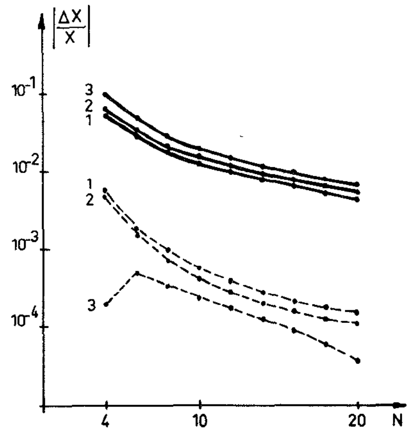

Fig. 2. Relative errors on shunt impedance versus the number of modes in the waveguide for the capacitive semidiaphragm. Continuous lines correspond to usual modal analysis, dotted lines to our method. The numbers 1, 2, 3 correspond to the normalized frequencies $k_0 b/\pi = 1.0198$, 1.414, and 1.9895 ($k_0 = \omega/c$).

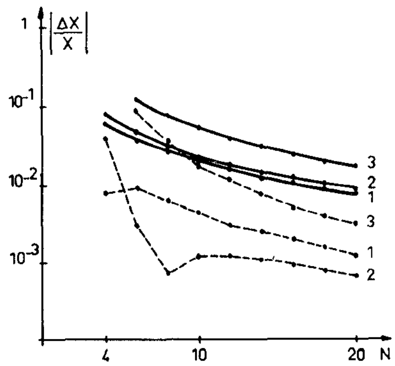

Fig. 3. Relative errors for inductive semidiaphragms for the same frequencies as Fig. 2.

III. Dielectric Steps

In order to explore the domain of interest of our method, we turn now towards the quite different problem of dielectric steps in parallel plate waveguide (Fig. 4). For $z > 0$ the guide is loaded with an isotropic slab of dielectric constant ε. In the literature such a junction has been especially studied in the case of TE excitation [6], [7]. The TM case is more interesting with respect to the edge effect.

The reader will find the main part of the calculation in Appendix B. We met two difficulties in this problem.

1) The series of asymptotic terms are very complicated [(B5)] and they cannot be summed with accuracy. We merely added the 100 first terms for each of them. Such a procedure applied

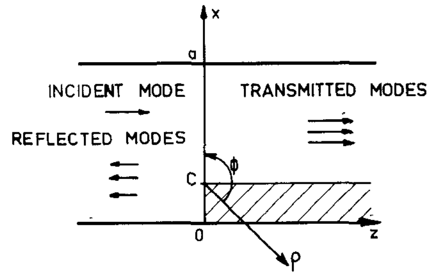

Fig. 4. Partially dielectric-filled junction in parallel-plate waveguide. The incident mode is the dominant mode, either in TM case (TEM mode) or in TE case.

to the diaphragm problem would have degraded the precision of the results.

2) How to appreciate the validity of the results and their convergence? Checking how the basic equations (1) and (2) are verified seems to be the best criterion. For that we calculate the ratios

$$\varepsilon_E = \frac{\int |E_L - E_R|^2 \, ds}{\int |E_L|^2 \, ds} \qquad \varepsilon_H = \frac{\int |H_L - H_R|^2 \, ds}{\int |H_L|^2 \, ds}$$

where the integrals are taken over the cross section, and where E_L, H_L and E_R, H_R are the computed transverse fields in the junction plane. (These fields would appear directly in (1) and (2) after truncation.)

Similar criteria have already been used [6], [8]. The ratios ε_E and ε_H must be small if the calculation is good. Unfortunately, the integrals are very difficult to calculate in our case because they contain infinite series due to the asymptotic terms, and these series behave quite badly (roughly as $\sum_k k^{-t} \cos k\pi x/a$ for the electric field and as $\sum_k k^{-1-t} \cos k\pi x/a$ for the magnetic field, with $0 < t < 1$). We retained only a rough approximation of them by adding the 100 first terms.

However, that criterion concerns the synthesis of the entire field while often only the first scattering coefficients have practical importance; there is no tight connection between the accuracy on these coefficients and the smallness of ε_E and ε_H. Thus if we are interested only in these coefficients (for instance, if we want to establish some equivalent circuit of the junction), it is better to observe the apparent convergence of these coefficients.

We have reported some of our results in Table I. Successively, one reads the number of modes in the empty waveguide, ε_E, with the usual method and with ours, ε_H with the usual method and with ours, and the reflection coefficient of the dominant mode R_1 with the usual method and with ours. The table corresponds to parameters $c/a = 0.231$ and $\varepsilon = 2.47$; the normalized frequency is $k_0 a/\pi = 0.191$ for the upper part, and 0.835 for the lower part. At first glance we may state the following conclusions.

1) The usual modal analysis gives scattering coefficients which converge rapidly. The rate of convergence decreases with higher frequency or higher dielectric constant, but it remains good (for instance, with $\varepsilon = 9.91$, $k_0 a/\pi = 0.835$, and 15 modes, one obtains R_1 with likely less than 0.2-percent error).

2) The usual modal analysis gives a good enough approximation for the magnetic field (low ε_H), but a poor one for the electric field (high ε_E), which was expected since only the electric field is singular. Both ε_E and ε_H increase with higher frequency, as is usual in modal analysis.

3) Our method gives some improvement on ε_E and ε_H, but the gain is rather moderate, especially on ε_E: that would signify that the singularity is not adequately described by the only asymptotic terms we retained in our calculation.

4) Our method seems to increase the rate of convergence for the scattering coefficients. However, we must point out two objections which cannot be inferred from the table. First, the results are sensitive to the accuracy with which the asymptotic series is summed. We noted that in the diaphragm problems. In the dielectric step the series cannot be summed easily and thus one can induce some error. Second, for some values of N one may obtain abnormal results (high ε_E and ε_H, R_1 far from its limit) which disappear for the following values of N. We do not explain that instability; perhaps it may be related to the problem of the summation of the series? In any case it is a serious objection against a large use of our method, at least in that problem.

In conclusion, if there is interest only in the first scattering coefficients, we think it preferable to use the usual modal analysis which converges very rapidly in that problem (compare the aforementioned 0.2-percent error for 15 modes with the curves in Figs. 2 and 3), for the gain in convergence with our method is seriously counterbalanced by the difficulty of summing the series. If one is interested in the whole electromagnetic field, only our method takes into account the singularity; but we have seen that the improvement so obtained is not sufficient. That is likely connected with the weakness of the singularity (with $\varepsilon = 2.47$,

TABLE I
COMPARISON BETWEEN USUAL METHOD AND OURS FOR THE DIELECTRIC STEP

N	ε_E		ε_H		R_1	
	usual	ours	usual	ours	usual	ours
	$k_0 a/\pi = 0.191$					
5	$1.3\ 10^{-2}$	$5.4\ 10^{-4}$	$1.2\ 10^{-6}$	$6.4\ 10^{-9}$	0.037298 + i 0.003106	0.037292 + i 0.003181
10	$7.4\ 10^{-2}$	$5.5\ 10^{-4}$	$1.7\ 10^{-7}$	$6.4\ 10^{-9}$	0.037294 + i 0.003162	0.037293 + i 0.003185
15	$5.\ 10^{-3}$	$5.\ 10^{-4}$	$7.7\ 10^{-8}$	$6.4\ 10^{-9}$	0.037293 + i 0.003175	0.037293 + i 0.003184
	$k_0 a/\pi = 0.835$					
5	$4.7\ 10^{-2}$	$8.2\ 10^{-3}$	$1.9\ 10^{-5}$	$3.1\ 10^{-8}$	0.050561 + i 0.006421	0.050523 + i 0.006510
10	$3.5\ 10^{-2}$	$8.4\ 10^{-3}$	$3.1\ 10^{-6}$	$3.5\ 10^{-8}$	0.050527 + i 0.006861	0.050495 + i 0.006867
15	$2.8\ 10^{-2}$	$7.8\ 10^{-3}$	$1.5\ 10^{-6}$	$3.6\ 10^{-8}$	0.050528 + i 0.006957	0.050532 + i 0.007011

close to the edge, one must go 160 times nearer the edge in order to double the electric field); the modal coefficients must not be represented by their asymptotic form below very high order.

For completeness, let us mention the case of TE excitation. There is no longer any edge effect in it [9], and no information can be deduced from it.[2] The usual modal analysis converges quite rapidly: for instance, we obtain $\varepsilon_E = 3.8\ 10^{-7}$ and $\varepsilon_H = 5.7\ 10^{-7}$ with $c/a = 0.2786$, $\varepsilon = 2.47$, $k_0a/\pi = 1.617$, and $N = 15$.

IV. Conclusion

Our method is an attempt to take into account a part of the information which is lost with the truncation in usual modal analysis. Such information is available when there is an edge effect. Thus one obtains more accurate results for the same number of modes. The interest of our method depends on two factors.

1) The field must be perturbed by the singularity on a large scale. If the field varies as r^p near the edge, the exponent p has to be as far as possible from zero. With a weak perturbation the ordinary modal analysis converges rapidly and our method is not worthy.

2) The series which appears in our method must be summed easily. This is possible with problems of diaphragms or bifurcation; it is not the case in dielectric steps.

Thus our method has little interest for dielectric steps. Nevertheless we gave some details on that problem because there are few published results on it. On the contrary, we obtain good results with diaphragms: over 99 percent of the useful frequency range in rectangular waveguide, we have less than 0.3-percent error in the inductive case with 20 modes (against 1.8 percent with the usual method). In the capacitive case, where the perturbation is very strong, we have less than 0.02-percent error with 20 modes (against 0.7 percent with the usual method).

Appendix A
Asymptotic Behavior of Modal Amplitudes

In order to make clearer the derivation of the function $f(k)$ from the edge condition, let us begin by the example of the capacitive diaphragm in parallel-plate waveguide [$b \to \infty$ in Fig. 1(a)]. It is well known that one may write the electric field as $e(x) + \bar{R}(x - a/2)^{-1/2}$ in the diaphragm plane [4, pp. 18–20], where $e(x)$ is regular and where $\bar{R}$ is an unknown constant. Thus the kth reflection coefficient is

$$R_k = \frac{2}{a}\int_{a/2}^{a}\left(e + \frac{\bar{R}}{\sqrt{x - a/2}}\right)\cos k\pi\frac{x}{a}\,dx.$$

When $k \to \infty$, the contribution from $e(x)$ behaves as k^{-2}, and the contribution from the singular term may be written as

$$\left[\frac{2}{\sqrt{ak\pi}}\int_0^{\infty}\frac{\cos(X + k\pi/2)}{\sqrt{X}}\,dX\right]\bar{R} = f(k)\bar{R}.$$

It decreases as $k^{-1/2}$ and thus is the dominant term.

We see that the asymptotic behavior of the modal coefficient is ruled by the expression of the field near the edge. That may be generalized as follows. One may always consider the modal coefficients as Fourier coefficients of the transverse fields; it is well known that the asymptotic behavior of Fourier coefficients of a given function is ruled by the singularity of highest order which appears in that function or its derivatives. In our case of singularity due to the edge effect, we can know exactly the form of the singularity (at least in simple cases—as far as we know there is no theory of edge effect for obstacles of arbitrary shape) and then we derive from it the asymptotic behavior of modal coefficients.

[2] On that point we do not understand [6, p. 275, condition f. 5].

Appendix B

The equations of the scattering of the TEM mode are

$$\sum_{k=0}^{\infty}(\delta_{k0} - R_k)\gamma_k\cos k\pi\frac{x}{a} = \sum_{p=1}^{\infty}T_p\frac{\Gamma_p}{\varepsilon(x)}h_p(x) \quad \text{(B1)}$$

$$\sum_{k=0}^{\infty}(\delta_{k0} + R_k)\cos k\pi\frac{x}{a} = \sum_{p=1}^{\infty}T_ph_p(x) \quad \text{(B2)}$$

where the unknown coefficients are R_k and T_p. The other quantities are

$$\gamma_k = (1 - (k\pi/ak_0)^2)^{1/2}, \quad (k = 0,1,2,\cdots)$$

$$\varepsilon(x) = \varepsilon(0 < x < c), \quad \text{or } 1(x > c)$$

$$h_p(x) = \begin{cases}\cos\alpha_p'x, & (0 < x < c)\\ A_p\cos\alpha_p(a - x), & (c < x < a)\end{cases}$$

with

$$A_p = \cos\alpha_p'c/\cos\alpha_pb$$

$$\alpha_p'^2 - \alpha_p^2 = k_0^2(\varepsilon - 1)$$

$$\alpha_p'\tan\alpha_p'c + \varepsilon\alpha_p\tan\alpha_pb = 0$$

$$\Gamma_p = (\varepsilon - (\alpha_p'/k_0)^2)^{1/2}.$$

We solved this system by taking the cross products of (B1) and (B2) with the functions $\{\cos k\pi(x/a)\}$ $(k = 0,1,\cdots)$, and then by eliminating the R_k.

In the vicinity of the edge, outside the dielectric, we may write the electric field as [9]

$$E_\rho = l_2\cos t(\varphi - \pi)\rho^{t-1} + l_1\sin\tau(\varphi - \pi)\rho^{\tau-1} \quad \text{(B3)}$$

where (ρ,φ) are the polar coordinates defined in Fig. 4, l_1, l_2 are unknown coefficients, and where $\tau - 1 = 1 - t$ with

$$\cos t\frac{\pi}{2} = \frac{1}{2}\frac{\varepsilon - 1}{\varepsilon + 1}, \quad (0 < t < 1). \quad \text{(B4)}$$

With respect to the symmetry plane of the dielectric wedge, the two terms of (B3) correspond to the symmetrical part and the antisymmetrical of the electric field. Both will be present since this plane is not a symmetry element of the entire problem. They lead to the following asymptotic form for the T_p:

$$-i\alpha_p'\left[\frac{c}{2\varepsilon} + \frac{b}{2}\left(1 + \left(\frac{1}{\varepsilon^2} - 1\right)\sin^2\alpha_p'c\right)\right]T_p$$

$$\simeq\cdots\left[-\cos\frac{3\pi t}{4}\cos\left(\alpha_p'c - \frac{\pi t}{2}\right) + \cos\frac{\pi t}{4}\cos\left(\alpha_p'b - \frac{\pi t}{2}\right)\right](\alpha_p')^{-t}\Gamma(t)l_2$$

$$+\left[-\cos\frac{3\pi t}{4}\cos\left(\alpha_p'c + \frac{\pi t}{2}\right) + \cos\frac{\pi t}{4}\cos\left(\alpha_p'b + \frac{\pi t}{2}\right)\right](\alpha_p')^{t-2}\Gamma(2 - t)l_1 \quad \text{(B5)}$$

where $\Gamma(z)$ is the gamma function. One sees that the antisymmetric field (l_1) corresponds to terms decreasing roughly as p^{-3+t}, while the symmetric part (l_2) corresponds to terms as p^{-1-t}. As (B4) gives t near unity, especially for low ε, one cannot neglect the antisymmetric part. We retained the two kinds of terms in our calculation.

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Improved Accuracy for Commensurate-Line Synthesis

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Abstract—**By employing a simple transformation that preserves numerical accuracy, improved precision is obtainable using a Richards' extraction technique to obtain characteristic impedances of commensurate transmission-line structures. Furthermore, reduced sensitivity to coefficient truncation can result in computational savings.**

In this short paper, we would like to report on certain aspects of the numerical calculation of characteristic impedances of cascaded commensurate-line networks. If correctly employed, Richards' extraction can be an extremely simple yet powerful and well-behaved algorithm. If misapplied, it can create large numerical inaccuracies.

The most important application of commensurate-line synthesis has been to problems of insertion loss design. The prescribed problem is typically the realization of a transducer gain function

$$s_{21}(\lambda)s_{21}(-\lambda) = \left.\frac{(1-\lambda^2)^n}{P_n(-\lambda^2)}\right|_{\lambda=j\Omega} = |s_{21}(j\Omega)|^2 \qquad (1)$$

where P_n is an even polynomial of degree $2n$ and the transformed frequency variable is

$$\Sigma + j\Omega = \lambda = \tanh p\tau, \qquad p = \sigma + j\omega$$

where ω is radian frequency.[1] For example Levy [1] gives commensurate-line characteristic impedances for low-pass Chebychev filter transducer gain functions. We here consider the numerical synthesis of arbitrary (but realizable) transducer gain functions, for example low-pass or bandpass filters, broad-band transformers, delay lines with amplitude selectivity, equalizers, matching networks, etc.

Considering a lossless reciprocal 2 port in the λ domain, the unitary requirement demands that the resistively terminated 2 port have an input reflection factor $s_{11}(\lambda)$ satisfying

$$s_{11}(\lambda)s_{11}(-\lambda) = 1 - s_{21}(\lambda)s_{21}(-\lambda)|_{\lambda=j\Omega} = |s_{11}(j\Omega)|^2.$$

The function $s_{11}(\lambda)$ is found by choosing the appropriate numerator and denominator root factors. The denominator must be Hurwitz but there is generally considerable flexibility in the choice of numerator roots, as well as a choice of a $\pm$ sign. Our task is to consider a suitable numerical method to determine the characteristic impedances of the structure, once $s_{11}(\lambda)$ has been given. We can of course proceed directly to the use of Richards' theorem for extracting the lines, given $s_{11}(\lambda)$ the input reflection factor of the resistively terminated cascade. However, this can lead to large numerical errors.

Our technique is to numerically operate on the input reflection factor $s_{11}(\lambda)$ of the resistively terminated cascade in a manner that preserves numerical accuracy and yields the reflection factor of the cascade of lines terminated in a short or open circuit rather than in a resistance. We have found that synthesizing this lossless function by Richards' theorem is superior numerically to making the line extraction calculations directly on $s_{11}(\lambda)$. In the latter case we deal with a two-element-kind network, whose input impedance is complex at real frequencies. If we use the lossless reflection factor, we deal with a purely reactive unit element network.

Separate the numerator and denominator terms of $s_{11}(\lambda)$ into even and odd parts

$$s_{11}(\lambda) = \frac{h_e(\lambda) + h_o(\lambda)}{g_e(\lambda) + g_o(\lambda)}. \qquad (2)$$

The reflection factors of the unterminated cascade are then

$$s_{1s} = \frac{(g_o + h_o) - (g_e - h_e)}{(g_o + h_o) + (g_e - h_e)} \qquad (3)$$

$$s_{1o} = \frac{(g_e + h_e) - (g_o - h_o)}{(g_e + h_e) + (g_o - h_o)} \qquad (4)$$

$$s_{2s} = \frac{(g_o + h_o) - (g_e + h_e)}{(g_o + h_o) + (g_e + h_e)} \qquad (5)$$

$$s_{2o} = \frac{(g_e - h_e) - (g_o - h_o)}{(g_e - h_e) + (g_o - h_o)}. \qquad (6)$$

Here s_{1s} and s_{1o} are the reflection factors at port 1 when the opposite port is short or open circuited, respectively, and s_{2s} and s_{2o} are similarly defined at port 2. These relations are generally valid starting with any resistively terminated reactance 2 port, e.g., interdigital filters, but must be slightly modified if $s_{12}(0) =$

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[1] $\lambda = \coth p\tau$ may be used equally well.